A REMARK ON THE HILBERT PARADOX OF BOLTZMANN EQUATION

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§1. **Introductions.** The behavior of dilute gas is described nicely by the following non-linear integro-differential equation due to L. Boltzmann,

$$D[p]=B[p\otimes p] \tag{1}$$

where $p(\boldsymbol{x}, \boldsymbol{v}, t)$ is the molecular density at the point $\boldsymbol{x}=(x_1, x_2, x_3)$ with velocity $\boldsymbol{v}=(v_1, v_2, v_3)$ at time t,

$$D[p]\equiv\frac{\partial p}{\partial t}+\sum_{i=1}^{3} v_i\frac{\partial p}{\partial x_i}+\sum_{i=1}^{3} a_i\frac{\partial p}{\partial v_i}\ ,$$

$\boldsymbol{a}=(a_1, a_2, a_3)$ being the acceleration of a molecule caused by the external force field, and $B[p\otimes q]$ is a symmetric bilinear functional defined by some integral on the variable $\boldsymbol{v}$ alone. The rigorous mathematical treatment of the initial value problem of (1) is so difficult that only a little result has been got ever since. However, D. Hilbert [2] proposed a nice procedure to find a class of solutions of (1), though purely a formal one, by expanding the solution p of (1) to a power series of some parameter. His recipe is the following.

Let p be expanded to a power series of some parameter ε,

$$p=p_0+\varepsilon p_1+\varepsilon^2 p_2+\cdots\ . \tag{2}$$

Putting this in the parameterized Boltzmann equation

$$D[p]=\frac{1}{\varepsilon}B[p\otimes p]\ , \tag{3}$$

we compare the coefficient of ε^n, $n=-1, 0, 1, 2, \cdots$. For ε^{-1} we get the following integral equation for p_0

$$B[p_0\otimes p_0]=0\ . \tag{4}$$

Using the explicit form of $B[p_0\otimes p_0]$, it is concluded that (4) is equivalent to p_0 being a locally Maxwellian distribution density function

$$p_0(\boldsymbol{x}, \boldsymbol{v}, t)=a\cdot\exp\left[-c\,\|\boldsymbol{v}-\boldsymbol{b}\|^2\right]\ , \tag{5}$$

where $a>0$, $\boldsymbol{b}=(b_1, b_2, b_3)$, $c>0$ are arbitrary functions of $\boldsymbol{x}$ and t. For ε^n $n\geq 0$, (3) is equivalent to

$$D[p_n]=\sum_{i=0}^{n+1} B[p_i\otimes p_{n+1-i}]$$

or

$$B[p_0 \otimes p_{n+1}] = \frac{1}{2}\left\{D[p_n] - \sum_{i=1}^{n} B[p_i \otimes p_{n+1-i}]\right\} . \tag{6}$$

Putting $p_{n+1} = p_0 \cdot \varphi_{n+1}$ and $p_0^{-1} \cdot B[p_0 \otimes (p_0 \cdot \varphi_{n+1})] = C[\varphi_{n+1}]$, (6) can be written as

$$C[\varphi_{n+1}] = \frac{1}{2} p_0^{-1} \cdot \left\{D[p_n] - \sum_{i=1}^{n} B[p_i \otimes p_{n+1-i}]\right\} . \tag{7}$$

From (5) $C[\varphi_{n+1}]$ turns out to be a nice linear integral operator for φ_{n+1} in $\boldsymbol{L}^2(p_0 \cdot d\boldsymbol{v})$, the null space of which is spanned by the five independent functions θ_j $j=1\sim5$ of $\boldsymbol{v}$, where $\theta_1 = 1$, $\theta_2 = v_1$, $\theta_3 = v_2$, $\theta_4 = v_3$, $\theta_5 = v_1^2 + v_2^2 + v_3^2$. If we suppose $p_0, p_1, \cdots, p_n$ are kown, Fredholm alternative can be applied, therefore, to the integral equation (7) for φ_{n+1}, and we conclude that (7) has a solution if and only if the right hand side of (7) is orthogonal to the null space, *i.e.*

$$\left(D[p_n] - \sum_{i=1}^{n} B[p_i \otimes p_{n+1-i}], \theta_j\right) = 0 \qquad j = 1\sim5 ,$$

where (,) denotes the inner product in $\boldsymbol{L}^2(d\boldsymbol{v})$. From the fact that

$$(B[f \otimes g], \theta_j) = 0 \qquad j = 1\sim5$$

for any f and g, the above condition is equivalent to

$$(D[p_n], \theta_j) = 0 \qquad j = 1\sim5 . \tag{8}$$

Supposing the vanishing boundary condition on p_n at $\|\boldsymbol{v}\| = +\infty$, (8) is transformed, integrating by parts, to an ordinary differential equation for five functions $p_n^{(j)} \equiv (p_n, \theta_j)$ $j=1\sim5$ of $\boldsymbol{x}$ and t. This equation can be solved uniquely, if we give the initial values $p_n^{(j)}(\boldsymbol{x}, 0) \equiv h_n^{(j)}(\boldsymbol{x})$. $p_n^{(j)}$ $j=1\sim5$ are called the hydrodynamical moments of p_n and the differential equation (8) to determine them is called the hydrodynamical equation in the language of hydrodynamics. For $n=0$, determining $p_0^{(j)}$ $j=1\sim5$ is equivalent to determining a, $\boldsymbol{b}$, c. So a, $\boldsymbol{b}$, c, and hence p_0 is determined uniquely if we give the five functions $h_0^{(j)}(\boldsymbol{x})$ $j=1\sim5$. If we suppose $p_0, p_1, \cdots, p_n$ are determined to satisfy (8), the integral equation (7) has the general solution of the following form

$$\varphi_{n+1} = \bar{\varphi}_{n+1} + \sum_{j=1}^{5} d_{n+1}^{(j)} \cdot \theta_j ,$$

where $\bar{\varphi}_{n+1}$ is a special solution of (7) and $d_{n+1}^{(j)} = d_{n+1}^{(j)}(\boldsymbol{x}, t)$ $j=1\sim5$ are are undetermined arbitrary functions of $\boldsymbol{x}$ and t alone, not depending on $\boldsymbol{v}$. Determining $d_{n+1}^{(j)}$ $j=1\sim5$ is equivalent to determing $p_{n+1}^{(j)}$ $j=1\sim5$. So the hydrodynamical equation (8) for $n+1$, which is the Fredholm condition for the existence of φ_{n+2}, determines p_{n+1} uniquely if we give the initial values $h_{n+1}^{(j)}(\boldsymbol{x})$ of five hydrodynamical moments of them.

Continueing this procedure inductively, we can determine all p_n $n=0, 1, 2, \cdots$, if we give the initial values of five hydrodynamical moments of them. In this way we can get a power series solution (2) for the equation (3). We call this solution the *Hilbert solution.* If we put $\varepsilon=1$ for the Hilbert solution, we get a solution of the original Boltzmann equation (1). At a glance, it seems that to get a Hilbert solution by the above procedure, we need infinitely many functions $h_n^{(j)}(\boldsymbol{x})$ $j=1\sim 5$, $n=0, 1, 2, \cdots$ of $\boldsymbol{x}$, but Hilbert insisted that the Hilbert solution $p(\boldsymbol{x}, \boldsymbol{v}, t)$ is determined by giving, in fact, only the initial values of the five hydrodynamical moments $p^{(j)}\equiv(p, \theta_j)$ $j=1\sim 5$ of p itself. His reasoning is as follows.

The Hilbert solution p is a power series solution of (3), for which the initial values of the hydrodynamical moments $p^{(j)}$ are given as

$$p^{(j)}(\boldsymbol{x}, 0)=h_0^{(j)}(\boldsymbol{x})+\varepsilon h_1^{(j)}(\boldsymbol{x})+\varepsilon^2 h_2^{(j)}(x)+\cdots \qquad j=1\sim 5\,. \tag{9}$$

From the above reasoning we know there is only one power series solution of (3) satisfying (9). We can construct the same solution by another way, as follows.

We take the same procedure as above, but for the initial values of the hydrodynamical moments $p_n^{(j)}$ of each p_n, we put

$$p_0^{(j)}(\boldsymbol{x}, 0)=h_0^{(j)}(\boldsymbol{x})+\varepsilon h_1^{(j)}(\boldsymbol{x})+\varepsilon^2 h_2^{(j)}(\boldsymbol{x})+\cdots$$

and

$$p_n^{(j)}(\boldsymbol{x}, 0)\equiv 0 \qquad \forall n\geq 1\,.$$

Supposing the well-posedness of the hydrodynamical equation (8), the new solutions $\tilde{p}_n$ $n=0, 1, 2, \cdots$ got by this procedure are all regular power series of ε, so

$$\tilde{p}=\tilde{p}_0+\varepsilon\tilde{p}_1+\varepsilon^2\tilde{p}_1+\cdots$$

is also a regular power series solution of (3). But as is clear by the above construction, the initial values of the hydrodynamical moments of $\tilde{p}$ are equal to those of p. Therefore by the uniqueness of the power series solution, we conclude that $p=\tilde{p}$. This means that for each fixed ε we can get the Hilbert solution of (3) if we know the the initial values of the five hydrodynamical moments of p itself, without needing to know the initial values of the hydrodynamical moments of all p_n $n=0, 1, 2, \cdots$. However it is natural to think that to determine the solution of (3) from its initial value, we need, not only the information on the five hydrodynamical moments of the initial value, but the whole information on the initial value. So we can assume that the class of all Hilbert solutions constitutes a very special class among all the solutions of the equation (3). We call this phenomenon on the Hilbert solutions the *Hilbert paradox,* as Uhlenbeck and Ford [3] called it.

In this paper we consider the problem on the Hilbert solution from another point of view as follows.

Consider the equation (3) with the initial value

$$f(x, v)=f_0(x, v)+\varepsilon f_1(x, v)+\varepsilon^2 f_2(x, v)+\cdots .$$

In general its solution p has not the regular form (2). We seek the condition on f under which p becomes regular in ε for all t. By this approach we can treat at once the problem of the hydrodynamical equations of all stages, such as the existence of the solutions of the hydrodynamical equations.

All the above discussions are so far purely formal ones and rigorous mathematical justification is "up in the air" in McKean's language.

§ 2. McKean's result

Instead of the original Boltzmann equation, H. P. McKean [1] treated the following very simple equation in a rigous mathematical sense along Hilbert's idea, which was extended by H. Hauge [4] and R. Ellis [5].

$$\begin{cases} D[p]\equiv\dfrac{\partial p}{\partial t}+v\dfrac{\partial p}{\partial x}=p(x, -v, t)-p(x, v, t)\equiv B[p] & (10) \\ \qquad p(x, v, 0)=f(x, v) & (11) \end{cases}$$

where

$$p=p(x, v, t)\,, \qquad x\in R\,, \qquad v=+1 \quad \text{or} \quad -1\,, \qquad t\in R^+ .$$

This equation is a linear equation and it describes the behavior of the density of 1-*dim* gas molecule with the velocity ± 1, chaning its velocity, not by the intermolecular collision as in the Boltzmann's case, but by other independent random law. The part of McKean's result to which we refer in this paper is the following two theorems for the parameterized equation

$$\begin{cases} D[p]=\dfrac{1}{\varepsilon}B[p] & (12) \\ p(x, v, 0)=f(x, v)=f_0(x, v)+\varepsilon f_1(x, v)+\varepsilon^2 f_2(x, v)+\cdots . & (13) \end{cases}$$

THEOREM A. *The equation* (12) (13) *has a power series solution in the Hilbert's sense if and only if f satisfies*

$$-v\frac{\partial f}{\partial x}+\frac{1}{\varepsilon}B[f]=\sum_{m=1}^{\infty}\binom{\frac{1}{2}}{m}\varepsilon^{2m-1}\frac{\partial^{2m}f}{\partial x^{2m}} . \tag{14}$$

The condition (14) is equivalent to

$$-vf_{odd}=\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}\varepsilon^{2m+1}\frac{\partial^{2m+1}f_{even}}{\partial x^{2m+1}}$$

where

$$f_{odd}(x, v)=\frac{f(x, v)-f(x, -v)}{2}, \qquad f_{even}(x, v)=\frac{f(x, v)+f(x, -v)}{2}.$$

Especially $f_{0\,odd}=0$ and $f_0=f_{0\,even}$. In this case f_{even} plays the role of the hydrodynamical moment of f, so we can say that if f is a initial value function of a Hibert solution of (12) (13), f_{odd} and hence f is completely determined from the knownledge of f_{even} only. If f is determined, p is uniquely determined by the time evolution of (12), and we conclude that Hilbert solution of (12) (13) is determined uniquely if we know the hydrodynamical moment f_{even} of the initial value.

THEOREM B. *If we put $f_{even}=f_0$, not depeding on ε, and if f_0 is an entire function of exponential type ≤ 1, then the formal power series solution $p=p_0+\varepsilon p_1+\varepsilon^2 p_2+\cdots$ in the Hilbert's sense really converges to an entire function of type ≤ 1, and the formal procedure of Hilbert is rigorously justified for this case. Moreover p has the following expression*

$$p=\int_{|z|=R} e^{xz}\left[1-\frac{v}{\varepsilon z}\{(1+\varepsilon^2 z^2)^{1/2}-1\}\right]\exp\left[\frac{t}{\varepsilon}\{(1+\varepsilon^2 t^2)^{1/2}-1\}\right]\hat{f}_{even}(z)dz\,, \tag{15}$$

where $\hat{f}_{even}(z)$ is the Laplace transform of f_{even} and $R>0$ $R|\varepsilon|<1$.

In this thoorem McKean's mention on the Hilbert paradox is a little ambigous, so we supplement it in this paper.

§ 3. Supplement to McKean's result

The solution of the differential equation (12) (13) is expressed explicitly as follows for smooth rapidly decreasing function f,

$$p=e^{-t/\varepsilon}\int_{-\infty}^{+\infty} e^{i\varepsilon x}\frac{\sinh\left\{(1-\varepsilon^2 z^2)^{1/2}\dfrac{t}{\varepsilon}\right\}}{(1-\varepsilon^2 z^2)^{1/2}}\{-iv\varepsilon z+B+1\}\hat{f}(z)dz$$
$$+e^{-t/\varepsilon}\int_{-\infty}^{+\infty} e^{i\varepsilon x}\cosh\left\{(1-\varepsilon^2 z^2)^{1/2}\frac{t}{\varepsilon}\right\}\hat{f}(z)dz\,, \tag{16}$$

where $\hat{f}(z)$ is the Fourier transform of $f(x)$ and B is the operator defined in (10). This is easily proved by the direct calculation of $D[p]$.

Using this formula we have, by an argument a little formal, the following theorem which is equivalent to *Theorem A*.

THEOREM A′. *The solution p of the differential equation* (12) (13) *is regular in ε for all t if and only if the Fourier transform $\hat{f}$ of the initial value f satisfies the following equation*

$$(-iv\varepsilon z+B)\hat{f}(z)=\{(1-\varepsilon^2 z^2)^{1/2}-1\}\hat{f}(z) \tag{17}$$

which is equivalent to (14).

Proof. We rewrite (16) as follows.

$$p=e^{-t/\varepsilon}\int_{-\infty}^{+\infty} e^{izx}\frac{\sinh\left\{(1-\varepsilon^2z^2)^{1/2}\dfrac{t}{\varepsilon}\right\}}{(1-\varepsilon^2z^2)^{1/2}}\{-iv\varepsilon z+B+1-(1-\varepsilon^2z^2)^{1/2}\}\hat{f}(z)dz$$
$$+\int_{-\infty}^{+\infty} e^{izx}\exp\left[\frac{t}{\varepsilon}\{(1-\varepsilon^2z^2)^{1/2}-1\}\right]\hat{f}(z)dz\,. \tag{18}$$

As $t/\varepsilon\{(1-\varepsilon^2z^2)^{1/2}-1\}$ is regular in ε for all t and $\hat{f}(z)$ is regular in ε, the second term of the right hand side of (18) is regular in ε for all t. So we have proved the theorem if we notice the following lemma which is almost clear. The equivalence of (14) and (17) is trivial by the formula of the Fourier transform.

LEMMA. *Let $k(z)=k_0(z)+\varepsilon k_1(z)+\varepsilon^2k_2(z)+\cdots$ be a regular power series in ε. In order that*

$$e^{-t/\varepsilon}\int_{-\infty}^{+\infty} e^{izx}\frac{\sinh\left\{(1-\varepsilon^2z^2)^{1/2}\dfrac{t}{\varepsilon}\right\}}{(1-\varepsilon^2z^2)^{1/2}}k(z)dz$$

is regular in ε for all $t\geq 0$, it is necessary and sufficient that $k(z)\equiv 0$.

Next we give the following simple generalization of McKean's *Theorem B.*

THEOREM B′. *Let $\{f_n(x)\}_{n=0,1,2,\cdots}$ be a set of entire functions of exponential type ≤ 1 satisfying*

$$\overline{\lim_{m\to\infty}}\sup_n\sup_{x\in K}\left|\frac{\partial}{\partial x^m}f_{n\,even}(x)\right|^{1/m}\leq 1$$

for any compact set $K\subset \boldsymbol{R}$, then for $|\varepsilon|<1$ the formal power series solution in the Hilbert's sense really converges to an entire function of exponential type ≤ 1, and the formal precedure of Hilbert is justified for this case. Moreover p has the same expression as (15).

The proof is almost clear from McKean's argument. To see the Hilbert paradox in this equation, we note the transformation $f_{even}\to\hat{f}_{even}$ is linear and preserves the regularity in ε and all other integrands of (15) are regular in ε, so we can justify the Hilbert paradox in this case, i.e. $p=\tilde{p}$ where $\tilde{p}$ is the power series solution got from the same procedure as in the Boltzmann equation.

References

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